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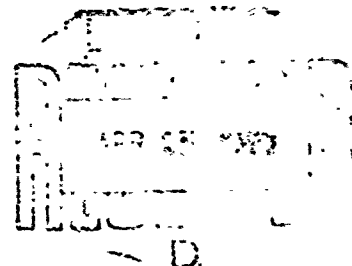
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13. ABSTRACT

This paper considers the question of why some convex quadratic programming algorithms fail and others succeed when applied to nonconvex quasi-convex quadratic programs. Several algorithms are identified as being capable of solving quasi-convex quadratic programs using only a finite number of arithmetic and logical operations. These algorithms are all primal feasible, pivot algorithms.

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ABSTRACT

This note considers the question of why some convex quadratic programming algorithms fail and others succeed when applied to nonconvex quasi-convex quadratic programs. Several algorithms are identified as being capable of solving quasi-convex quadratic programs using only a finite number of arithmetic and logical operations. These algorithms are all primal feasible, pivot algorithms.

Recent papers by Martos (1971) and Cottle and Ferland (1970a,b) examine the class of quadratic functions that are quasi-convex and pseudo-convex on the nonnegative orthant. Ferland (1971) studies the class of quadratic functions that are quasi-convex and pseudo-convex on convex sets possessing non-empty interiors.

A quadratic program of the form

(1a) $$\text{minimize } \varphi(x) = c^T x + \frac{1}{2} x^T D x$$

(1b) $$\text{subject to} \quad Ax \geq b$$

(1c) $$x \geq 0,$$

where the function $\varphi(x)$ is quasi-convex (pseudo-convex) on the set of "primal" feasible points $X = \{x \in R^n : Ax \geq b, x \geq 0\}$ is called a quasi-convex (pseudo-convex) quadratic program. There is no loss in generality in assuming D is a symmetric matrix and this is assumed through this paper. The Kuhn-Tucker conditions for (1) can be stated in the form

(2a) $$u = c + Dx - A^T y$$

(2b) $$v = -b + Ax$$

(2c) $$u \geq 0, \ x \geq 0, \ v \geq 0, \ y \geq 0$$

(2d) $$u^T x = 0. \quad v^T y = 0.$$

Specializing results obtained by Mangasarian (1965, or see 1969) to the case of a pseudo-convex quadratic program gives the following theorem.

<u>Theorem 1.</u> If the point $(\bar{u}, \bar{x}, \bar{v}, \bar{y})$ satisfies the Kuhn-Tucker conditions of a pseudo-convex quadratic program, then the point $\bar{x}$ is a solution of the quadratic program.

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For completeness, some of the properties of general quasi-convex and pseudo-convex functions will be repeated here. A more extensive discussion of general quasi-convex and pseudo-convex functions may be found in Mangasarian's (1969) recent book on nonlinear programming and recent results on these classes of functions can be found in Ferland (1971).

The following statements are equivalent when φ is defined on a convex set X:

(i) φ is a quasi-convex function on the set X,

(ii) for any α, the set $\{x \in X : \varphi(x) \leq \alpha\}$ is a convex set,

(iii) $\varphi(\alpha x^1 + (1-\alpha) x^2) \leq$ maximum $[\varphi(x^1), \varphi(x^2)]$ for x^1, $x^2 \in X$,

   $0 \leq \alpha \leq 1$, and .

(iv) if $\varphi(x^2) \leq \varphi(x^1)$ and φ is a continuously differentiable function, then $\nabla\varphi(x^1) (x^2 - x^1) \leq 0$.

A continuously differentiable function φ defined on a set X is pseudo-convex if for all x^1, $x^2 \in X$, $\nabla\varphi(x^1) (x^2 - x^1) \geq 0$ implies $\varphi(x^2) \geq \varphi(x^1)$. If a function is continuously differentiable on a convex set X and it is a convex function, then it is a pseudo-convex function on X; if it is pseudo-convex on X, then it is quasi-convex on X.

Quasi-convex and pseudo-convex quadratic functions are very closely related. Martos (1971, Theorem 3) showed that if $\varphi(x) = c^T x + \frac{1}{2} x^T D x$ is not convex but is quasi-convex on the nonnegative orthant (R_+^n) and the matrix $\begin{pmatrix} D & c \\ c^T & 0 \end{pmatrix}$ has no row of zeros, then φ is pseudo-convex on the set $R_+^n - \{0\}$, which is the nonnegative orthant with the origin removed. After Martos obtained this result, Cottle and Ferland (1970b) proved the following theorem, which permits one to replace the origin.

Theorem 2. If the quadratic function $\varphi(x)$ is not convex but is quasi-convex on the nonnegative orthant, then it is pseudo-convex on the nonnegative orthant provided $c \neq 0$.

Martos (1969) shows that when $c = 0$, φ can be pseudo-convex on R_+^a only if it is convex on R^a.

A reasonable computational test to determine if a quadratic function is quasi-convex on the nonnegative orthant can be based on the following theorem characterizing quasi-convex quadratic functions given by Cottle and Ferland (1970a).

Theorem 3. The quadratic function $\varphi(x) = c^T x + \frac{1}{2} x^T D x$ is not convex, but is quasi-convex on R_+^a if and only if

(a)
$$\begin{pmatrix} D & c \\ c^T & 0 \end{pmatrix} \leq 0$$

and

(b) the matrix $\begin{pmatrix} D & c \\ c^T & 0 \end{pmatrix}$ has only one negative eigenvalue.

The following result, also due to Cottle and Ferland (1970a), is the basis for a finite sufficiency test. A finite test is an algorithm requiring only a finite number of arithmetic and logical operations to determine if an object (function) possesses a particular property.

Theorem 4. A quadratic function is not convex but is pseudo-convex on R_+^a if

(a)
$$\begin{pmatrix} D & c \\ c^T & 0 \end{pmatrix} \leq 0$$

and

(b) the matrix $\begin{pmatrix} D & c \\ c^T & 0 \end{pmatrix}$ has negative leading principal minors.

The satisfactions of conditions (a) and (b) of both theorems 3 and 4 can be determined using standard techniques from numerical linear algebra. These techniques require only a finite number of arithmetic

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operations. Since it is possible to identify many pseudo-convex and quasi-convex quadratic programs using a finite test, one would like to solve them using a finite algorithm, many of which are available for solving <u>convex</u> quadratic programs. Some of these algorithms may fail when applied to a pseudo-convex quadratic program. Martos (1971) shows by example that Wolfe's simplex method for quadratic programming (1959) is such an algorithm and identifies the Frank-Wolfe algorithm (1956, Section 6) as an algorithm that can be used to solve pseudo-convex quadratic programs when the set X of primal feasible points is compact. However, the Frank-Wolfe algorithm is not finite. After giving an example of the use of the Frank-Wolfe algorithm to solve a pseudo-convex quadratic program Martos makes the following statement.

> "Challenged by the finiteness of many convex quadratic programming methods we, of course, made several attempts to find a finite method for the quasi-convex case, too. With no success in this direction one should address himself to the question: how and why might a method fail? In this respect we have only a partial answer to the first part (how?) of the question. To this end, we can show a counterexample, where the application of the well known quadratic simplex method of Wolfe demonstrates how this one can fail. Other methods may presumably fail otherwise."

Contrary to the impression one would have after reading the above statement, there exist several finite algorithms for solving pseudo-convex quadratic programs. The oldest and best known of these methods is Beale's quadratic programming algorithm (1955, or see Beale (1959) or (1967)). Another method is Ritter's algorithm for finding a local minimum (1966, also see Cottle and Mylander (1970)). Two more recent methods are those by Keller (1969) and Mylander (1971).

Keller's method (1969) is a modification of the Dantzig-Cottle principal pivoting algorithm for solving linear complementarity

problems (1967, also see Cottle (1968)). The Kuhn-Tucker conditions for a quadratic program are a special case of a linear complementarity problem. Hence Keller's method is applied to the Kuhn-Tucker conditions stated earlier in (2). If necessary, a Phase I procedure is performed to find a point satisfying (2a,b) such that $x \geq 0$ and $v \geq 0$. That is, a primal feasible solution is found. In Phase II, primal feasibility is maintained while a solution satisfying all the Kuhn-Tucker conditions is sought After finding a primal feasible point, the algorithm can terminate in only two ways—either with a solution to the Kuhn-Tucker conditions or with an indication that the objective function φ is not bounded from below on the feasible set.

Mylander's algorithm (1971) is a modification of Lemke's algorithm (1965, 1968) for solving linear complementarity problems. As with Keller's algorithm, a Phase I procedure is first applied, if necessary, to find a primal feasible point. Then, in what is called the "positive phase," the covering vector prescribed by Lemke (1965, 1968) is replaced with one that has positive entries covering only the u and y variables that are basic in (2a,b). The other elements of the covering vector are zero. Using this type of covering vector ensures that the rules of Lemke's algorithm generate primal feasible points at each step. The positive phase terminates either with an indication that the objective function φ is not bounded below on the primal feasible set X or with a solution to the Kuhn-Tucker conditions. Mylander's modification of Lemke's also has a negative phase which is used to seek additional solutions of the Kuhn-Tucker conditions of nonconvex quadratic programs,

but the use of the negative phase is not necessary in processing a convex or pseudo-convex quadratic program.

For a pseudo-convex program any solution of the Kuhn-Tucker conditions gives a solution to the programming problem. For quadratic programming problems involving the minimization of functions that are quasi-convex but not pseudo-convex on the nonnegative orthant the point $x = 0$, if feasible, is a stationary point. That is, there exist u, v, y, with $x = 0$ satisfying the Kuhn-Tucker conditions (2). In this case $x = 0$ is either a saddle point or a maximizing point. Theorem 2 indicates this case can occur only when $c = 0$. Algorithms that will solve pseudo-convex programs can be used to solve quasi-convex programs by perturbing the c vector by a small amount; some of the zero elements of c being replaced by small negative numbers.

The common feature of all the algorithms listed earlier for solving quasi-convex quadratic programs is they are primal feasible algorithms. They require a starting point in the feasible set X and generate points in the feasible set X.

Another common feature of the finite algorithms for quasi-convex quadratic programming is they are pivot algorithms that maintain basic solutions to a set of linear equations. With the exception of Beale's algorithm, the linear equations are the Kuhn-Tucker equations (2a,b) or an augmentation of the Kuhn-Tucker equations. These algorithms work by entering a non-basic variable into the basis in place of a basic variable in seeking to satisfy all the Kuhn-Tucker conditions. Pivot algorithms for quadratic programming can terminate in only one of four ways:

(i) with a solution to the Kuhn-Tucker conditions,

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(ii) with a non-basic variable specified to enter the basis, but no basic variable specified to leave the basis, this is called termination on a ray,

(iii) neither member of the (basic variable, non-basic variable) interchange being specified by the rules of the algorithm, or

(iv) the specified pivot element specified by a basic variable and a non-basic variable to be interchanged being of the wrong sign or zero.

The second case, termination on a ray, corresponds to the form of termination occurring in the simplex method for linear programming when there is an unbounded solution. That is, the non-basic variable can be assigned any positive value and all the basic variables remain nonnegative.

To determine if a pivot algorithm for convex quadratic programs can be used to solve quasi-convex quadratic programs it is necessary to show that the third and fourth forms of termination cannot occur and that the second form of termination, termination on a ray, indicates either the objective function is unbounded below on the primal feasible set X or the set X is empty. If the algorithm works with primal feasible points, then termination on a ray must be interpretable as an indication that the objective function is not bounded from below on X.

Beale's algorithm is an adaptation of the method of steepest descent that exploits the fact that the derivatives of a quadratic function are linear. If it terminates on a ray, the objective must go to minus infinity on that ray. When it terminates because the rules do not specify a pivot and if the basic solution at hand is nondegenerate,

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then there is no small feasible move that will decrease the objective function. Farkas' lemma then can be used to show the existence of values for u and y satisfying the Kuhn-Tucker conditions. The forth form of termination cannot arise in Beale's algorithm.

The principal pivot algorithm of Dantzig and Cottle cannot be counted on to solve quasi-convex programs because termination on a ray cannot be interpreted for this class of problems. Also, this algorithm is predicated on the expectation the main diagonal of the matrix of coefficients of the non-basic variables, when on the same side of the equality sign as the constant column, contains only nonnegative elements after the completion of a major cycle. However, Keller has modified the rules of the principal pivot algorithm for linear complementarity problems arising from the Kuhn-Tucker conditions of quadratic programs to handle the case of negative elements and to find and maintain the feasibility of the primal variables. Using the fact that only primal feasible points are generated he is able to show termination on a ray indicates the objective function is not bounded below on X without any assumption on the nature of the quadratic form of the objective function.

Lemke's algorithm makes no structural assumption relative to the matrix of coefficients of the linear equations to which it is applied. It can be applied to the Kuhn-Tucker equations arising from a non-convex quadratic program and termination will occur after a finite number of pivots either on a ray or with a solution to the Kuhn-Tucker conditions. However, for nonconvex quadratic programs termination on a ray cannot be interpreted. Mylander (1971) gives an example of Lemke's

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algorithm terminating on a ray for a quasi-convex quadratic program possessing a finite solution on a compact feasible set X. Mylander's modification of Lemke's algorithm makes it possible to guarantee that a solution to Kuhn-Tucker conditions resulting from a quasi-convex program will be found or if termination on a ray occurs then the objective function is not bounded from below on X.

Ritter's algorithm can be viewed as an extension of Houthakker's quadratic programming algorithm (1960, also see van de Panne and Whinston (1966)) to produce local minima for nonconvex quadratic programs. Houthakker's algorithm can fail when applied to a quasi-convex program because a pivot element expected to be positive might r·t be positive. If the problem is a convex quadratic program it can be proved that the required pivotal element must always be positive (van de Panne and Whinston (1966)). Ritter extended the algorithm by providing additional rules to handle the case of a nonpositive element that is the desired pivot element in Houthakker's algorithm.

In summary, any programming algorithm converging to a point where the Kuhn-Tucker conditions are satisfied or giving an indication of the occurrence of an objective function that is unbounded below on the primal feasible set X can be used to solve quasi-convex quadratic programs. Such an algorithm must not require the assumption that the quadratic form be positive semi-definite to prove that the algorithm does not stop prematurely. Nor can it make use of the assumption the objective function is convex on the feasible set to prove that termination on a ray indicates there is no solution to the Kuhn-Tucker conditions. There are several known finite quadratic programming algorithms meeting these

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requirements and they can be used to process quasi-convex quadratic programs. The common features of the known finite algorithms that can be used to solve quasi-convex programs are that they are primal feasible, pivot algorithms.

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